

Finite Geometry and the Radon Transform

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Finite geometry is used to underpin operators acting in finite, d , dimensional Hilbert space. Quasi distribution and Radon transform underpinned with finite dual affine plane geometry (DAPG) are defined in close analogy with the continuous ($d \rightarrow \infty$) Hilbert space case. An essential role in these definitions play projectors of states of mutual unbiased bases (MUB) and their Wigner function-like mapping onto the generalized phase space that lines and points of DAPG constitutes.

I. INTRODUCTION

In this article we consider, in a d -dimensional Hilbert space, the underpinning of information theoretic related operators with finite geometry.

The general idea of such a relation was studied intensely, [1, 3–6]. Our approach is focused on a particular branch of finite geometrical system, [7–9, 26]: dual affine plane geometry (DAPG) whose main features are outlined in section III. The operators that we discuss are the operators involved in "Mutual Unbiased Bases" (MUB) [7, 8, 19, 22]. The study is confined to dimensionality, $d=p$, a prime, $\neq 2$. We itemize the relevant definitions/properties concerning these operators in the next section, section II, in our attempt to have this paper self contained.

The general idea of geometrically underpinned Hilbert space is as follows. A Geometry is defined by interrelation among points and lines. Points are considered the fundamental entities and lines are aggregates of points. We specify the rules for the so called (finite) affine plane geometry (APG) and its dual (DAPG) in section III. A specific arrangement of points and lines abiding by such rules form a realization of the geometry. The association of Hilbert space operators with geometrical points and lines define a geometrical underpinning of the Hilbert space operators. Consistency of an underpinning is attested to by the consistency of the operators implied physical results/relation and the interrelations dictated by the geometry. The present study relates to the so called state reconstruction [9, 16]: the Hilbert space MUB state projectors allow the *diagonal* elements (of all MUB states) of the density operator to define in full the density operator. We illustrate this with a study of Radon transform [27, 28] which, hitherto, was confined to studies in continuum. Consistency may here be checked by state reconstruction via the inverse transform.

The focal issue in the present analysis is finite dimensional Radon transform. This transform, for the continuous case ($d \rightarrow \infty$) [27, 28], is widely used both in down to earth application [30–32] mathematical studies [33, 34], and state reconstruction analysis [11, 13, 35, 36]. In the present work, dealing with finite dimensional Radon transform, it is, much like in the continuous case, analyzed via quasi distribution, viz Wigner-like function defined here in the finite phase space-like points and lines of DAPG.

The Radon transformation [16, 27, 29] involves angular variables and thus its formulation in finite (Hilbert) space dimensionality is somewhat intricate. We overcome this by adopting a "physical" approach whose rationale, within the continuum, $d \rightarrow \infty$ case is as follows. The Wigner function, $W_A(q, p)$ maps an operator, \hat{A} in Hilbert space onto a c -number function in phase space [17, 42]. When the operator is the state density operator, $\hat{\rho}$, the resultant Wigner function, $W_\rho(q, p)$, is quasi distribution, i.e. it enjoys many attributes of a distribution (it may, however, become negative) [11, 13, 14, 41]. A particularly attractive attribute that underscores its role as quasi distribution is its marginal with respect to the position q . Thus integrating along a vertical line (the p coordinate) for a fixed $q=x'$: $\int \frac{dqdp}{2\pi} \delta(x' - q) W_\rho(q, p)$, gives the probability for the system in a state ρ (i.e. whose quasi distribution is $W_\rho(q, p)$) to have its position value x' . In other words this integral equals the expectation value of the projector $|x'\rangle\langle x'|$ that projects the eigen state, $|x'\rangle$, of the operator \hat{x} . The Wigner phase space mapping of this projector is $W_{|x'\rangle\langle x'|}(q, p) = \delta(x' - q)$. This continues to hold when we replace the position operator by arbitrary MUB state projector, $|x', \theta\rangle\langle \theta, x'|$, with $|x', \theta\rangle$ the eigenfunction of $\hat{X}_\theta = \hat{x}\cos\theta + \hat{p}\sin\theta$, eigen value x' [11, 13, 14]. In this case, since the Wigner function $W_{|x', \theta\rangle\langle \theta, x'|}(q, p) = \delta(x' - q\cos\theta - p\sin\theta)$, the integral is along the line $y = -q\sin\theta + p\cos\theta$. Thus here too the marginal relates to MUB projectors [1, 16, 45] and we have for the probability, $P(x', \theta : \rho)$, of the state ρ being found in $x'\theta\rangle\langle \theta, x'|$ ($C = \cos\theta$, $S = \sin\theta$),

$$P(x', \theta : \rho) \equiv \text{tr}(\rho |x', \theta\rangle\langle \theta, x'|) = \int \frac{dqdp}{2\pi} \delta(x' - Cq - Sp) W_\rho(q, p). \quad (1)$$

In this form we recognize the marginal probability of the MUB projector's expectation value as the Radon transform

of the quasi distribution, $W_\rho(q, p)$. An explicit account is given in section II. We use these observations as our guide for the definition of the finite dimensional Radon transform. Indeed the inversion of the transform is, in effect, a state reconstruction: it gives the state's quasi distribution, in terms of the probabilities $P(x', \theta : \rho)$.

In the finite dimensional ($d=p$, a prime $\neq 2$) Hilbert space we define a Wigner function-like mappings of operators onto lines and points of DAPG. We show that our Wigner function-like mapping of the density operator, $V_\rho(j)$, onto DAPG lines, L_j , - has all the quasi distribution attributes possessed by $W_\rho(q, p)$ in the $d \rightarrow \infty$ case. The marginals of $V_\rho(j)$ that give the probabilities of the system being in some eigen function of the finite dimensional MUB sets are now given as summation are along the DAPG line, L_j these being assured via the function $\Lambda_{\alpha, j}$, (defined in the text, α designates a DAPG point and j a line.) which corresponds to the delta function in the $d \rightarrow \infty$. Moreover, the marginals here, much like the continuous case, are informationally complete i.e. they allow the reconstruction of the corresponding (quasi) distribution which, in turn, [11, 16] determine the state. Thus our finite dimensional Radon transform is the marginals of the quasi distributions pertaining to the projectors of MUB states in complete analogy with the $d \rightarrow \infty$ case.

The invertibility attribute of the transform allows the extraction of the system's state from lines' summations that is informationally complete. For clarity we give a brief explicit review of this for the continuous case in section II. The finite geometry with which we underpin the theory, viz dual affine plane geometry (DAPG) is outlined in section III. MUB is defined in section IV. Section V contains the basic underpinning theory. Here and in the succeeding section we list some useful formulae. The formulation of the finite dimension Radon transform with its explicit inversion as well as the definition our quasi distribution are contained in the succeeding section, section VII. Section VIII introduces underpinning of finite dimensional Hilbert space operators with affine plane geometry, APG. In the last section, section IX, we summarize and discuss the results.

II. MUTUALLY UNBIASED BASES AND RADON TRANSFORM - THE $d \rightarrow \infty$ CASE.

The rationale of our analysis is the relation between informationally complete measurements which we enumerate via mutual unbiased bases (MUB) and the sought after state of the system. In this section we outline the (known e.g. [11]) approach for the continuous case thereby clarifying, we hope, the (known, [11, 13]) result that the state, actually its Wigner representative function, is determined from the *diagonal* elements of the density operator with respect to (all) the states in all the MUB bases. To this end we briefly review the notion of MUB their measurements and the extraction thereof the Wigner function of the density operator.

Mutual unbiased bases, MUB, in concept were introduced by Schwinger [10] in his studies of vectorial bases for Hilbert spaces that exhibit "maximal degree of incompatibility". The eigenvectors of \hat{x} and \hat{p} , $|x\rangle$ and $|p\rangle$ respectively are example of such bases. The information theoretical oriented appellation "mutual unbiased bases" were introduced by Wootters [8]. Two complete, orthonormal vectorial bases, \mathcal{B}_1 , \mathcal{B}_2 , are said to be MUB if and only if $(\mathcal{B}_1 \neq \mathcal{B}_2)$

$$\forall |u\rangle, |v\rangle \in \mathcal{B}_1, \mathcal{B}_2 \text{ resp.}, |\langle u|v\rangle| = \text{const.} \quad (2)$$

i.e. the absolute value of the scalar product of vectors from *different* bases is independent of the vectorial label within either basis. This implies that if a state vector is measured to be in one of the states, $|u\rangle$, of \mathcal{B}_1 it is equally likely to be in any of the states $|v\rangle$ of any other MUB, \mathcal{B}_2 . (The value of the $|\langle u|v\rangle|$ may depend on the *bases*, $\mathcal{B}_1, \mathcal{B}_2$, which indeed is the case for the limit $d \rightarrow \infty$, the continuous dimensionality.) MUB are found to be of interest in several fields. The ideas are useful in a variety of cryptographic protocols [12] and signal analysis [2].

We now outline, for the continuous Hilbert space dimensionality, some salient MUB features [16, 19, 43, 45]. Consider the so called quadrature [11, 13, 14] operator \hat{X}_θ and its eigen states state $|x', \theta\rangle$,

$$\hat{X}_\theta |x', \theta\rangle = x' |x', \theta\rangle. \quad (3)$$

We now show that the states, $|x', \theta\rangle$ form a complete orthonormal basis for the (rigged) Hilbert space, with the vectors labeled by x' and the basis by θ , and bases with different θ , $0 \leq \theta \leq \pi$, are MUB. We begin by noting, that, as can be easily checked [11, 13, 19]

$$\hat{X}_\theta = U^\dagger(\theta) \hat{x} U(\theta) = C \hat{x} + S \hat{p}, \quad (4)$$

where

$$U(\theta) = e^{-i\theta\hat{a}^\dagger\hat{a}}; \quad \hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \quad (5)$$

and $C = \cos\theta$, $S = \sin\theta$. Thus the solution to Eq(3) may be written in terms of the eigenvectors of the position operator [16, 19, 43–45],

$$|x, \theta\rangle = U^\dagger(\theta)|x\rangle. \quad (6)$$

This defines our phase choice [15]. We now use the well known result for the harmonic oscillator propagator, [16, 40], to get the x representation solution,

$$\langle x'|x; \theta\rangle = \frac{1}{\sqrt{-i2\pi S}} e^{-\frac{1}{2S}([x^2 + x'^2]C - 2xx')}. \quad (7)$$

This form ascertains [16] that $\lim_{\theta \rightarrow 0} \langle x'|x; \theta\rangle = \delta(x - x')$, and $\lim_{\theta \rightarrow \pi/2} \langle x'|x; \theta\rangle = e^{xx'}/\sqrt{2\pi}$. This phase differs from the more common one [7] and leads to an expression that is symmetric $x \leftrightarrow x'$, a property that facilitates several calculations. We now verify, by direct calculation, that the bases $\{|x; \theta\rangle\}$ and $\{|x'; \theta'\rangle\}$ with $\theta \neq \theta'$ are MUB:

$$|\langle x'; \theta'|x; \theta\rangle| = |\langle x'|U^\dagger(\theta - \theta')|x\rangle| = \frac{1}{\sqrt{2\pi|S(\theta, \theta')|}}, \quad (8)$$

$S(\theta, \theta') = \sin(\theta - \theta')$. Thus the number $|\langle x'; \theta'|x; \theta\rangle|$ is independent of the vectorial labels x, x' . We used the relation $U(\theta')U^\dagger(\theta) = U^\dagger(\theta - \theta')$ and Eq(6).

The Wigner function that represents an arbitrary operator \hat{A} in phase space [11, 13, 14, 17, 41, 42] is given by,

$$W_A(q, p) = \int dy e^{ipy} \langle q - y/2 | \hat{A} | q + y/2 \rangle. \quad (9)$$

Thus the Wigner function maps an Hilbert space operator to a c-number function in phase space. An important attribute of this mapping is

$$\text{tr} \hat{A} \hat{B} = \int \frac{dq dp}{2\pi} W_A(q, p) W_B(q, p). \quad (10)$$

It can be shown [11, 13] that the Wigner function of the density operator is real though *not* non negative (in general this, with Eq.10), instigates its classification as a *quasi* distribution in phase space. This completes our brief review of MUB and quasi distributions for continuous Hilbert space [19]. Now we turn to the state reconstruction and Radon transform role for this case.

The probability of obtaining x' upon measuring the quadrature operator \hat{X}_θ , Eq.(4) for the state ρ is, [11, 13, 16]

$$\rho(\theta, x') \equiv \text{tr}(\rho |x'; \theta\rangle \langle \theta, x'|) = \int \frac{dq dp}{2\pi} W_\rho(q, p) W_{|x', \theta\rangle}(q, p), \quad (11)$$

where we used Eq.(10) to write the RHS of the equation. Explicit evaluating $W_{X_\theta}(q, p)$, via Eq.(9), gives for $\rho(\theta, x')$,

$$\rho(\theta, x') = \int \frac{dq dp}{2\pi} W_\rho(q, p) \delta(x' - Cq - Sp). \quad (12)$$

This identifies [32] $\rho(\theta, x')$ as the Radon transform of $W_\rho(q, p)$. Thus the Radon transform [28, 29] $\mathcal{R}f(L)$ of a function $f(x, y)$ is the integral of the function over straight line, L . The line in Eq.(12) is a line in phase space whose given by $x' = Cq + Sp$. (The full transform requires the integration over all parallel lines i.e. over all values of x'). Solving for $W_\rho(q, p)$ in terms of the observables $\rho(\theta, x')$ [28]:

$$W_\rho(q, p) = \mathcal{R}^{-1}(\mathcal{R}W_\rho)(q, p) = \mathcal{R}^{-1}\rho(\theta, x'). \quad (13)$$

Here \mathcal{R} represents Radon transform. Direct calculation gives [11, 13, 16, 19]

$$W_\rho(q, p) = -\frac{1}{\pi} \int_0^\pi d\theta \mathcal{P} \int_{-\infty}^\infty dx' \frac{\partial \rho(x, \theta / \partial x}{x - qC - pS}. \quad (14)$$

Thus inverting the Radon transformation is state reconstruction as the state ρ is determined once its Wigner representative is known [11].

Note that $\rho(\theta, x')$ plays a double role: (a) As a marginal it is required via Eq.(10), that the RHS of Eq.(11) contain the Wigner function of the MUB state projector, $|x', \theta\rangle \langle \theta, x'|$. Alternatively, (b), as the Radon transform of $W_\rho(q, p)$, this is consistent because the Wigner function of the projector is a delta function assuring, in the integration, that x' equals $Cq + Sp$ in phase space.

III. FINITE GEOMETRY AND HILBERT SPACE OPERATORS

We now briefly review the essential features of finite geometry required for our study [1, 24, 26, 37, 38].

A finite plane geometry is a system possessing a finite number of points and lines. There are two kinds of finite plane geometry: affine and projective. We shall confine ourselves to affine plane geometry (APG) which is defined as follows. An APG is a non empty set whose elements are called points. These are grouped in subsets called lines subject to:

1. Given any two distinct points there is exactly one line containing both.
2. Given a line L and a point S not in L ($S \ni L$), there exists exactly one line L' containing S such that $L \cap L' = \emptyset$. This is the parallel postulate.

3. There are 3 points that are not collinear.

It can be shown [37, 38] that for $d = p^m$ (a power of prime) APG can be constructed (our study here is for $d=p$) and the following properties are, necessarily, built in:

- a. The number of points is d^2 ; S_α , $\alpha = 1, 2, \dots, d^2$ and the number of lines is $d(d+1)$; L_j , $j = 1, 2, \dots, d(d+1)$.
- b. A pair of lines may have at most one point in common: $L_j \cap L_k = \lambda$; $\lambda = 0, 1$ for $j \neq k$.
- c. Each line is made of d points and each point is common to $d+1$ lines: $L_j = \bigcup_{\alpha}^d S_{\alpha}^j$, $S_{\alpha} = \bigcap_{j=1}^{d+1} L_j^{\alpha}$.
- d. If a line L_j is parallel to the distinct lines L_k and L_i then $L_k \parallel L_i$. The d^2 points are grouped in sets of d parallel lines. There are $d+1$ such groupings.
- e. Each line in a set of parallel lines intersect each line of any other set: $L_j \cap L_k = 1$; $L_j \not\parallel L_k$.

The above items will be referred to by APG (x), with $x=a,b,c,d$ or e .

The existence of APG implies [24, 37, 38] the existence of its dual geometry DAPG wherein the points and lines are interchanged. Since we shall study extensively this, DAPG, we list the corresponding properties for it. We shall refer to these by DAPG(y):

- a. The number of lines is d^2 , L_j , $j = 1, 2, \dots, d^2$. The number of points is $d(d+1)$, S_{α} , $\alpha = 1, 2, \dots, d(d+1)$.
- b. A pair of points on a line determine a line uniquely. Two (distinct) lines share one and only one point.
- c. Each point is common to d lines. Each line contain $d+1$ points.
- d. The $d(d+1)$ points may be grouped in sets of d points no two of a set share a line. Such a set is designated by $\alpha' \in \{\alpha \cup M_{\alpha}\}$, $\alpha' = 1, 2, \dots, d$. (M_{α} contain all the points not connected to α - they are not connected among themselves.) i.e. such a set contain d disjoint (among themselves) points. There are $d+1$ such sets:

$$\begin{aligned} \bigcup_{\alpha=1}^{d(d+1)} S_{\alpha} &= \bigcup_{\alpha=1}^d R_{\alpha}; \\ R_{\alpha} &= \bigcup_{\alpha' \in \alpha \cup M_{\alpha}} S_{\alpha'}; \\ R_{\alpha} \cap R_{\alpha'} &= \emptyset, \alpha \neq \alpha'. \end{aligned}$$

- e. Each point of a set of disjoint points is connected to every other point not in its set.

A particular arrangement of lines and points that satisfies APG(x), $x=a,b,c,d,e$ is referred to as a realization of APG. Similar prescription holds for DAPG.

We now consider a particular realization of DAPG of dimensionality $d = p, \neq 2$ which is the basis of our present study. We arrange the aggregate the $d(d+1)$ points, α , in a $d \cdot (d+1)$ matrix like rectangular array of d rows and $d+1$ columns. Each column is made of a set of d points $R_{\alpha} = \bigcup_{\alpha' \in \alpha \cup M_{\alpha}} S_{\alpha'}$; DAPG(d). We label the columns by $b=-1, 0, 1, 2, \dots, d-1$ and the rows by $m=0, 1, 2, \dots, d-1$. (Note that the first column label of -1 is for convenience and does not designate negative value of a number.) Thus $\alpha = m(b)$ designate a point by its row, m , and its column, b ; when b is allowed to vary - it designate the point's row position in every column. We label the left most column by $b=-1$ and with increasing values of b , the basis label, as we move to the right. Thus the right most column is $b=d-1$. We now assert that the $d+1$ points, $m_j(b)$, $b = 0, 1, 2, \dots, d-1$, and $m_j(-1)$, that form the line j which contain the two (specific) points $m(-1)$ and $m(0)$ is given by (we forfeit the subscript j - it is implicit),

$$\begin{aligned} m(b) &= \frac{b}{2}(c-1) + m(0), \text{ mod}[d] \quad b \neq -1, \\ m(-1) &= c/2. \end{aligned} \tag{15}$$

The rational for this particular form is clarified in Section V. Thus a line j is parameterized fully by $j = (m(-1), m(0))$. We now prove that the set $j = 1, 2, 3, \dots, d^2$ lines covered by Eq.(15) with the points as defined above

form a DAPG.

1. Since each of the parameters, $m(-1)$ and $m(0)$, can have d values the number of lines d^2 ; the number of points in a line is evidently $d+1$. DAPG(a).
2. The linearity of the equation precludes having two points with a common value of b on the same line. Now consider two points on a given line, $m(b_1), m(b_2)$; $b_1 \neq b_2$. We have from Eq.(15), ($b \neq -1$, $b_1 \neq b_2$)

$$\begin{aligned} m(b_1) &= \frac{b_1}{2}(c-1) + m(0), \quad \text{mod}[d] \\ m(b_2) &= \frac{b_2}{2}(c-1) + m(0), \quad \text{mod}[d]. \end{aligned} \quad (16)$$

These two equation determine uniquely (for $d=p$, prime) $m(-1)$ and $m(0)$. DAPG(b).

For fixed point, $m(b)$, $c \Leftrightarrow m(0)$ i.e the number of free parameters is d (the number of points on a fixed column). Thus each point is common to d lines. That the line contain $d+1$ is obvious. DAPG(c).

3. As is argued in 2 above no line contain two points in the same column (i.e. with equal b). Thus the d points, α , in a column form a set $R_\alpha = \bigcup_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'}$, with trivially $R_\alpha \cap R_{\alpha'} = \emptyset$, $\alpha \neq \alpha'$, and $\bigcup_{\alpha=1}^{d(d+1)} S_\alpha = \bigcup_{\alpha=1}^d R_\alpha$. DAPG(d).

4. Consider two arbitrary points *not* in the same set, R_α defined above: $m(b_1), m(b_2)$ ($b_1 \neq b_2$). The argument of 2 above states that, for $d=p$, there is a unique solution for the two parameters that specify the line containing these points. DAPG(e).

We illustrate the above for $d=3$, where we explicitly specify the points contained in the line $j = (m(-1) = (1, -1), m(0) = (2, 0))$

$$\begin{pmatrix} m \backslash b & -1 & 0 & 1 & 2 \\ 0 & \cdot & \cdot & \cdot & (0, 2) \\ 1 & (1, -1) & \cdot & (1, 1) & \cdot \\ 2 & \cdot & (2, 0) & \cdot & \cdot \end{pmatrix}$$

For example the point $m(1)$ is gotten from

$$m(1) = \frac{1}{2}(2-1) + 2 = 1 \quad \text{mod}[3] \rightarrow m(1) = (1, 1).$$

IV. FINITE DIMENSIONAL MUTUAL UNBIASED BASES, MUB, BRIEF REVIEW

In a finite, d -dimensional, Hilbert space two complete, orthonormal vectorial bases, $\mathcal{B}_1, \mathcal{B}_2$, are said to be MUB if and only if ($\mathcal{B}_1 \neq \mathcal{B}_2$)

$$\forall |u\rangle, |v\rangle \in \mathcal{B}_1, \mathcal{B}_2 \text{ resp.}, \quad |\langle u|v\rangle| = 1/\sqrt{d}. \quad (17)$$

The physical meaning of this is that knowledge that a system is in a particular state in one basis implies complete ignorance of its state in the other basis.

Ivanovic [20] proved that there are at most $d+1$ MUB, pairwise, in a d -dimensional Hilbert space and gave an explicit formulae for the $d+1$ bases in the case of $d=p$ (prime number). Wootters and Fields [8] constructed such $d+1$ bases for $d = p^m$ with m an integer. Variety of methods for construction of the $d+1$ bases for $d = p^m$ are now available [2, 21, 23]. Our present study is confined to $d = p \neq 2$.

We now give explicitly the MUB states in conjunction with the algebraically complete operators [10, 19] set: \hat{Z}, \hat{X} . Thus we label the d distinct states spanning the Hilbert space, termed the computational basis, by $|n\rangle$, $n = 0, 1, \dots, d-1$; $|n+d\rangle = |n\rangle$

$$\hat{Z}|n\rangle = \omega^n |n\rangle; \quad \hat{X}|n\rangle = |n+1\rangle, \quad \omega = e^{i2\pi/d}. \quad (18)$$

The d states in each of the $d+1$ MUB bases [19, 21] are the states of computational basis and

$$|m; b\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} \omega^{\frac{b}{2}n(n-1) - nm} |n\rangle; \quad b, m = 0, 1, \dots, d-1. \quad (19)$$

Here the d sets labeled by b are the bases and the m labels the states within a basis. We have [21]

$$\hat{X}\hat{Z}^b|m;b\rangle = \omega^m|m;b\rangle. \quad (20)$$

For later reference we shall refer to the computational basis (CB) by $b=-1$. Thus the above gives $d+1$ bases, $b=-1, 0, 1, \dots, d-1$ with the total number of states $d(d+1)$ grouped in $d+1$ sets each of d states. We have of course,

$$\langle m;b|m';b\rangle = \delta_{m,m'}; \quad |\langle m;b|m';b'\rangle| = \frac{1}{\sqrt{d}}, \quad b \neq b'. \quad (21)$$

This completes our discussion of MUB.

V. GEOMETRIC UNDERPINNING OF MUB QUANTUM OPERATORS

We now consider a DAPG as underpinning a two sets of operators acting in a d -dimensional Hilbert space, these are

$$\hat{A}_\alpha; \quad \alpha = 1, 2, \dots, d(d+1), \quad \hat{P}_j; \quad j = 1, 2, \dots, d^2.$$

Here \hat{A}_α are associated with the $d(d+1)$ points, S_α while \hat{P}_j are associated with the d^2 lines, L_j . We now define interrelation among the operators of the sets in a way that reflects the geometry,

$$\hat{A}_\alpha = \frac{1}{d} \sum_{j \in \alpha} \hat{P}_j. \quad (22)$$

These entail, using DAPG(a,b,d),

$$\hat{P}_j = \sum_{\alpha \in j} \hat{A}_\alpha - \frac{1}{d} \sum_{j'} \hat{P}_{j'}. \quad (23)$$

We now list some known finite dimensional features. These will allow its underpinning with DAPG which are presented in the succeeding Section.

We consider $d=p$, a prime. For $d=p$ we may construct $d+1$ MUB [1, 7, 20, 21]. Returning to our state labeling we label the MUB states by $|m, b\rangle$. We designate the computational basis, CB, by $b=-1$, while $b = 0, 1, 2, \dots, d-1$ labels the eigenfunction of, resp. XZ^b . m labels the state within a basis. (Note that assigning $b=-1$ to the CB is for reference only.) The projection operator defined by,

$$\hat{A}_\alpha \equiv |m, b\rangle\langle b, m|; \quad \alpha = \{b, m\}; \quad b = -1, 0, 1, 2, \dots, d-1; \quad m = 1, 2, \dots, d. \quad (24)$$

The point label, $\alpha = (m, b)$ is now associated with the projection operator, A_α . We now consider a realization, possible for $d=p$, a prime, of a d dimensional DAPG, as points marked on a rectangular whose width (x-axis) is made of $d+1$ column of points, each column is labelled by b , and its height (y axis) is made of d points each marked with m . The total number of points is $d(d+1)$ - there are d points in each of the $d+1$ columns. We associate the d points $m = 1, 2, \dots, d$, in each set labelled by b S_α ; $\alpha \sim (m, b)$ to the *disjointed* points of DAPG(d), viz. for fixed b $\alpha' \in \alpha \cup M_\alpha$ form a column. The columns are arranged according to their basis label, b . The first being $b=-1$, $\alpha_{-1} = (m, -1)$; $m = 0, 1, \dots, d-1$, signifies the computational basis (CB). The lines are now made of $d+1$ points each of different b . To a line L_j we associate an operator \hat{P}_j . Now DAPG(c) (and Eq.(22)) implies:

$$\begin{aligned} \sum_m^d |m, b\rangle\langle b, m| &= \sum_{\alpha' \in \alpha \cup M_\alpha}^d \hat{A}_{\alpha'} = \hat{I} \\ \sum_{\alpha}^{d(d+1)} \hat{A}_\alpha &= (d+1)\hat{I}. \end{aligned} \quad (25)$$

Returning to Eq.(23) we have,

$$\sum_j^{d^2} \hat{P}_j = d\hat{I}.$$

$$\hat{P}_j = \sum_{\alpha \in j}^{d+1} \hat{A}_\alpha - \hat{I}. \quad (26)$$

These imply

$$\text{tr}\{\hat{A}_\alpha \hat{P}_j\} = \Lambda_{\alpha,j}; \quad \Lambda_{\alpha,j} = \begin{cases} 1, & \alpha \in j, \\ 0, & \alpha \ni j. \end{cases} \quad (27)$$

To prove Eq.(27) we note Eq.(21, 26) to write,

$$\begin{aligned} \hat{A}_\alpha \hat{P}_j &= \hat{A}_\alpha + \sum_{\alpha' \neq \alpha} \hat{A}_{\alpha'} \hat{A}_\alpha - \hat{A}_\alpha \quad \alpha \in j \\ &= \sum_{\alpha' \neq \alpha} \hat{A}_{\alpha'} \hat{A}_\alpha - \hat{A}_\alpha \quad \alpha \ni j, \end{aligned} \quad (28)$$

taking the trace implies (27). This result, Eq.(27), involves *geometrical factors only*. We note that the $\Lambda_{\alpha,j}$ may equally be viewed as

$$\text{tr}\{\hat{A}_\alpha \hat{P}_j\} = \Lambda_{\alpha,j}; \quad \Lambda_{\alpha,j} = \begin{cases} 1, & j \in \alpha, \\ 0, & j \ni \alpha. \end{cases} \quad (29)$$

The proof for this case is given in Appendix A.,

e.g. for d=3 the underpinning's schematics is (the choice of c/2 for the CB vectors will prove convenient below).

$$\begin{pmatrix} m \setminus b & -1 & 0 & 1 & 2 \\ 0 & A_{(c/2=0,-1)} & A_{(0,0)} & A_{(0,1)} & A_{(0,2)} \\ 1 & A_{(c/2=1,-1)} & A_{(1,0)} & A_{(1,1)} & A_{(1,2)} \\ 2 & A_{(c/2=2,-1)} & A_{(2,0)} & A_{(2,1)} & A_{(2,2)} \end{pmatrix}$$

The geometrical line, $L_j, j = (1, 2)$ given above (end of Section III) upon being transcribed to its operator formula is via Eq.(26),

$$P_{j(c/2=1, m_0=2)} = A_{(c/2=1,-1)} + A_{(2,0)} + A_{(1,1)} + A_{(0,2)} - \hat{I}. \quad (30)$$

Returning to Eqs.(24,19), these equations imply that, the projection operators A_α , in the CB representation are given by,

$$(A_{\alpha=m,b})_{n,n'} = \begin{cases} \frac{\omega^s}{d}; & s = (n - n')(\frac{b}{2}[n + n' - 1] - m), \quad b \neq -1, \\ \delta_{n,n'} \delta_{c/2,n} & b = -1. \end{cases} \quad (31)$$

First we argue that every $(A_{\alpha=m,b})_{n,n'}$ in column b has in every other column b' $b \neq b'$, $b, b' \neq -1$ one projector such that $(A_{\alpha=m,b})_{n,n'} = (A_{\alpha'=m',b'})_{n,n'}$: Consider two distinct columns, b, b' ($b, b' \neq -1$) and given the matrix elements n, n' ($n \neq n'$) of a projector $(A_{\alpha=m,b})_{n,n'}$, compare it with $(A_{\alpha'=m',b'})_{n,n'}$. If s (Eq.(31)) is $\neq s'$ i.e. $\frac{b}{2}(n + n') - 1 - m \neq \frac{b'}{2}(n + n') - 1 - m'$ pick another projector in the same column, b' (i.e vary m'). Since $m' = 0, 1, 2, \dots, d-1$ there is one (and only one) $(A_{\alpha'=m',b'})_{n,n'}$ such that $(A_{\alpha=m,b})_{n,n'} = (A_{\alpha'=m',b'})_{n,n'}$. Now consider another matrix element $(A_\alpha)_{\bar{n}, \bar{n}'}$. We have trivially that $(A_\alpha)_{\bar{n}, \bar{n}'} = (A_{\alpha'})_{\bar{n}, \bar{n}'}$ iff $\bar{n} + \bar{n}' = n + n'$. i.e. all matrix elements (n, n')

with $n+n'=c$ (constant) are such that $(A_\alpha)_{n,n'} = (A_{\alpha'})_{n,n'}$. These elements are situated along a line perpendicular to the diagonal of the matrices. We refer to this perpendicular as FV (foliated vector), it is parameterized by c . We now assert that all other (non diagonal) matrix elements are unequal. i.e. for $b \neq b'$, $(A_{\alpha=m,b})_{n,n'} \neq (A_{\alpha'=m',b'})_{n,n'}$, $\forall n, n' \ni FV$. Proof: Let two elements n, n' and l, l' with $n \neq n'$; $l \neq l'$ in the two matrices be equal. Thus ($c=n+n'$, $c'=l+l'$):

$$\begin{aligned} \frac{b}{2}(c-1) - m &= \frac{b'}{2}(c-1) - m', \quad \text{and} \\ \frac{b}{2}(c'-1) - m &= \frac{b'}{2}(c'-1) - m', \end{aligned} \quad (32)$$

These *imply* $c=c'$, QED. Now consider $s=0$. Then all the matrix elements along FV are $1/d$. We have then that for $(A_\alpha)_{n,n'} = (A_{\alpha'})_{n,n'} = \omega^s/d$, $s \neq 0$, $d-1$ matrix elements along FV are all distinct. The diagonal is common to all. We have, thus, a prescription for d projectors, $A_{(m,b)}$, one for each b , ($b \neq -1$), all having equal matrix elements along FV labelled by c . We supplement these with the projector $A_{(c/2,-1)} = |c/2\rangle\langle c/2|$ to have the $d+1$ "points" constituting a line j . ($|c/2\rangle$ being a state in the CB.) Thus our line is formed as follows: It emerges from $A_{(c/2,-1)}$ continues to $A_{(m(0),0)}$ in the $b=0$ column. Then it continues to the points $A_{m(b),b}$ in succession: $b=1, 2, \dots, d-1$ with $m(b)$ determined by

$$\frac{b}{2}(c-1) - m(b) = \frac{b+1}{2}(c-1) - m(b+1).$$

Thus the two parameters, $c=2m(-1)$ and $m(1)$, determine the line i.e. $j=(m(-1), m(1))$. The general formula for the line is thus Eq.(15 now acquiring a meaning in terms of the point operators, $A_{\alpha=m(b),b}$. It is of interest that, if we associate the CB states with the position variable, q , of the continuous problem and its Fourier transform state, viz $b=0$ (cf. Eq.(19)), with the momentum, p , we have that the line of the finite dimension problem is parameterized with "initial" values of "q" and "p" i.e. $m(-1)$ and $m(0)$.

The discussion of the properties of the line thus defined confirm that these lines realize DAPG lines. The analysis above indicate that the line operator, $P_{j=(m(-1), m(0))}$. We now list some important consequences of this. We have shown that the matrix elements along a FV direction are the same for all the point operators $A_{\alpha \in j}$. Indeed that is how we defined our lines. On the other hand we argued that the matrix elements *not* along the FV are all distinct. Thence summing up d such terms residing on a fixed P_j (*excluding the $b=-1$ and the diagonal term*) sums up for each matrix element n, n' the d roots of unity for all matrix elements not on FV, hence for all c ,

$$\left(\sum_{\alpha \in j, \alpha \ni \alpha_{-1}}^d \hat{A}_\alpha - \hat{I} \right)_{n,n'} = 0; \quad n, n' \ni n + n' = c; \quad \alpha_{-1} = |c/2\rangle\langle c/2|. \quad (33)$$

Thus $(\hat{P}_j)_{n,n'} = (\sum_{\alpha \in j}^{d+1} \hat{A}_\alpha - \hat{I})_{n,n'} \neq 0$ *only* along FV, and is 1 along the diagonal at $c/2=m(-1)$. The sum over the matrix elements on a FV, which are the same for all the $\hat{A}_{\alpha \in j, \neq -1}$ simply cancel the $1/d$. We illustrate this for the example considered above Eq.(30), viz: $d=3$, line with $m(-1)=1$, $m(0)=2$, i.e. $j=(1,2)$ Evaluating the point operators, \hat{A}_α ,

$$A_{(c/2=1,-1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_{(2,0)} = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, A_{(1,1)} = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & 1 \\ \omega^2 & 1 & 1 \end{pmatrix}, A_{(0,2)} = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega \\ \omega^2 & \omega^2 & 1 \end{pmatrix}, \quad (34)$$

and evaluating the sum, Eq.(30), gives

$$P_{j:(m(-1)=1, m(0)=2)} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}. \quad (35)$$

Quite generally,

$$(P_{j=m(-1), m(0)})_{n,n'} = \begin{cases} \omega^{-(n-n')m(0)} \delta_{\{(n+n'), 2m(-1)\}} \\ 0 \text{ otherwise.} \end{cases} \quad (36)$$

Thus,

$$(\hat{P}_{j=c/2, m_0}^2)_{n, n'} = \delta_{n, n'}. \text{ i.e. } \hat{P}_j^2 = \hat{I} \forall j. \quad (37)$$

$$\text{Theorem : } \text{tr} \hat{P}_j \hat{P}_{j'} = d \delta_{j, j'}.$$

$$\begin{aligned} \text{Proof : } \text{Eq.(37)} &\rightarrow \text{tr} \hat{P}_j^2 = \text{tr} \hat{I} = d, \\ \text{Eq(26)} &\rightarrow \text{tr} \hat{P}_j \hat{P}_{j'} = \text{tr} \left(\hat{P}_j \sum_{\alpha \in j'} \hat{A}_\alpha \right) - \text{tr} \hat{P}_j \\ \text{Eq(27)} &\rightarrow \text{tr} \hat{P}_j \hat{A}_{\beta \in j} - 1 = 0, \quad j' \neq j. \end{aligned}$$

Where we used $\text{tr} \hat{P}_j = 1$ as follows from Eq.(37) and that two distinct lines share one point (see DAPG(b)).

VI. FURTHER ATTRIBUTES OF THE LINE OPERATOR, \hat{P}_j

The line operator, \hat{P}_j is hermitian being the sum of hermitian operators. Noting that, trivially, $\text{tr} \hat{P}_j = 1$ and $\hat{P}^2 = \hat{I}$, Eq(37) implies the spectral representation

$$(\hat{P}_j) = \begin{cases} \delta_{n, n'} & n = 1, 2, \dots, \frac{d+1}{2} \text{ (} \frac{d+1}{2} \text{ elements),} \\ -\delta_{n, n'} & n = \frac{d+3}{2}, \frac{d+5}{2}, \dots, d \text{ (} \frac{d-1}{2} \text{ elements.)} \end{cases} \quad (38)$$

We thus have that

$$\mathbb{P}_j \equiv \frac{\hat{P}_j + \hat{I}}{2} \quad (39)$$

is a projection operator onto the $\frac{d+1}{2}$ eigenstates of \hat{P}_j with eigenvalue +1. Recalling that $\hat{P}_j = \sum_{\alpha \in j} \hat{A}_\alpha - \hat{I}$ and $\hat{P}^2 = \hat{I}$ implies (what we term) the Fluctuation Distillation Formula (FDF):

$$\sum_{\alpha \neq \beta; \alpha, \beta \in j} \hat{A}_\alpha \hat{A}_\beta = \sum_{\alpha \in j} \hat{A}_\alpha. \quad (40)$$

The proof is given in Appendix B.

Summing both sides of the Eq.(27) over α and use Eq.(26) and Eq.(21) to write, for the LHS

$$\begin{aligned} \text{tr} \left[\left(\sum_{\alpha \in j} \hat{A}_\alpha \right) \left(\sum_{\alpha' \in j} \hat{A}_{\alpha'} - \hat{I} \right) \right] &= \text{tr} \left[\sum_{\alpha \neq \beta} \hat{A}_\alpha \hat{A}_\beta \right] \\ &= \text{tr} [(\hat{P}_j + \hat{I}) \hat{P}_j] = \text{tr} [\hat{P}_j + \hat{I}]. \end{aligned} \quad (41)$$

Where in the last step we used the relation, Eq.(37), $\hat{P}_j^2 = \hat{I}$ (valid for $A_{\alpha_{-1}} = |c/2\rangle\langle c/1|$).

To illustrate the spectral decomposition we consider again the line $j = \{m(-1) = 1, m(0) = 2\}$ in $d=3$. This line runs through the points (1,-1), (2,0), (1,1) and (0,2). The line operator $\hat{P}_{(1,2)}$ was given above,

$$P_{(1,2)} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & \omega/2 \\ 0 & 1 & 0 \\ \omega^2/2 & 0 & 1/2 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 & \omega/2 \\ 0 & 1 & 0 \\ -\omega^2/2 & 0 & 1/2 \end{pmatrix}. \quad (42)$$

Thus the matrix is diagonal in the orthonormal basis,

$$\begin{pmatrix} \omega/\sqrt{2} \\ 0 \\ \omega^2/\sqrt{2} \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} \omega/\sqrt{2} \\ 0 \\ -\omega^2/\sqrt{2} \end{pmatrix}.$$

The associated projection operator,

$$\mathbb{P}_{(1,2)} = \frac{\hat{P}_j + \hat{I}}{2} = \frac{1}{\sqrt{2}}[\omega^2|0\rangle + \omega|2\rangle] \frac{1}{\sqrt{2}}[\omega\langle 0| + \omega^2\langle 2|] + |1\rangle\langle 1|. \quad (43)$$

Let \mathbb{M}_j be a $d(d+1)$ matrix. Its $d+1$ columns are made of the d elements, $\langle n|m(b), b\rangle$ with $|m(b), b\rangle$ the state whose projection, $|m(b), b\rangle\langle b, m(b)|$, is the point of the line j in the column b . Define \mathbb{M}_j^\dagger as the d columns matrix whose rows are made up of $d+1$ columns corresponding to the adjoint of \mathbb{M}_j . For example, using Eq.(30) the line $j : c/2 = 1, m(0) = 2$ is given by the aggregate of points: $(c/2 = 1, m(0) = 2), (2, 0), (1, 1), (0, 2)$ giving as a normalized state line operator,

$$\mathbb{M}_j = \frac{1}{\sqrt{d+1}} \begin{pmatrix} 0 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1 & \omega/\sqrt{3} & \omega^2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \omega^2/\sqrt{3} & \omega^2/\sqrt{3} & \omega^2/\sqrt{3} \end{pmatrix}, \mathbb{M}_j^\dagger = \frac{1}{\sqrt{d+1}} \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{3} & \omega^2/\sqrt{3} & \omega/\sqrt{3} \\ 1/\sqrt{3} & \omega/\sqrt{3} & \omega/\sqrt{3} \end{pmatrix}.$$

While the line $j = c/2 = 0, m_0 = 0$: viz

$$(c/2 = 0, m(0) = 0), (0, 0), (1, 1), (2, 2) \Rightarrow \mathbb{M}_{(c/2=0, m(0)=0)} = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} & \omega^2/\sqrt{3} & \omega/\sqrt{3} \\ 0 & 1/\sqrt{3} & \omega^2/\sqrt{3} & \omega/\sqrt{3} \end{pmatrix}$$

The construction of the matrices assures

$$\begin{aligned} \text{tr} \mathbb{M}_j^\dagger \mathbb{M}_j &= 1 \\ \text{tr} \mathbb{M}_j^\dagger \mathbb{M}_{j'} &= \frac{1}{d+1} \quad j \neq j'. \end{aligned} \quad (44)$$

The proof is based on the DAPG attribute that two distinct lines share precisely one point, and for equal b distinct states are orthogonal. e.g.

$$\text{tr} \mathbb{M}_{(c/2=0, m(0)=0)}^\dagger \mathbb{M}_{(c/2=1, m(0)=2)} = \frac{1}{4}.$$

The normalized "line" matrix, \mathbb{M}_j , is the "square root" of the normalized "line" projection operator, viz:

$$\mathbb{M}_j \mathbb{M}_j^\dagger = \frac{2}{d+1} \mathbb{P}_j. \quad (45)$$

This is almost self evident: The RHS equals

$$\sum_{\alpha \in j} A_\alpha = \sum_b |m(b), b\rangle\langle b, m(b)|$$

While the the LHS is (for the n, n' matrix element)

$$\sum_b \langle n|m(b), b\rangle\langle b, m(b)|n'\rangle.$$

i.e. they are identical.

We now demonstrate that these line operators are geometric in origin. Thus we associate the line operator \mathbb{M}_j with $\tilde{\mathbb{M}}_j$, defined by the replacement, in the former, of every column b ($\neq -1$), as follows. Instead of the elements $\langle n|m(b), b\rangle$, $n = 0, 1, \dots, d-1$ with $m(b)$ a point in the line j , we have in the column b of $\tilde{\mathbb{M}}_j$, 1 at the row corresponding to $m(b)$. Thus in our example of $j=(1,2)$ for $d=3$ (cf. Eq.(30)), we have,

$$\mathbb{M}_{(1,2)} = \frac{1}{\sqrt{4}} \begin{pmatrix} 0 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1 & \omega/\sqrt{3} & \omega^2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \omega^2/\sqrt{3} & \omega^2/\sqrt{3} & \omega^2/\sqrt{3} \end{pmatrix} \Rightarrow \tilde{\mathbb{M}}_{(1,2)} = \frac{1}{\sqrt{4}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$\tilde{\mathbb{M}}_j$ is geometrical: it involves, in essence, the drawn line. We note that the transition between \mathbb{M}_j and $\tilde{\mathbb{M}}_j$ is "local" unitary transformation: we require a distinct unitary transformation for each column (basis), b .

VII. DAPG UNDERPINNED QUASI-DISTRIBUTIONS

We now define a Wigner function like quasi-distribution, $V_\rho[j = (m(-1), m(0))]$. It maps the density operator $\hat{\rho}$ onto the lines, $j = \{(m(-1), m(0))\}$, and points $\alpha = \{m(b)\}$, of DAPG. This, $V_\rho(j)$, Wigner function-like completely determine and is determined by the state of the system, $\hat{\rho}$. (An essentially equivalent definition relates to arbitrary operators in the d -dimensional Hilbert space.) We then consider summation of $V_\rho(j)$ along a fixed values of α that represent an MUB projector, as the finite dimensional Radon transform of $V_\rho(j)$. The inversion of this, viz. the reconstruction of $V_\rho(j)$ from summations along such points is our finite dimensional inversion of the Radon transform.

Returning to the line operators, \hat{P}_j , $j = 1, 2, \dots, d^2$, $j \equiv (m(-1), m(0))$, Eq.(15), we utilize their orthogonality, $\text{tr} \hat{P}_j \hat{P}_{j'} = d \delta_{j,j'}$, to write,

$$\hat{\rho} = \frac{1}{d} \sum_j^{d^2} (\text{tr} \hat{\rho} \hat{P}_j) \hat{P}_j = \frac{1}{d} \sum_j^{d^2} V_\rho(j) P_j. \quad (46)$$

Where we defined $V_\rho(j) \equiv \text{tr}(\hat{\rho} \hat{P}_j)$. A partial list of attributes of $V_\rho(j)$ is the following.

1. $V_\rho(j) = (V_\rho(j))^*$.
2. $\frac{1}{d} \sum_j^{d^2} V_\rho(j) = \text{tr} \rho (\frac{1}{d} \sum_j^{d^2} \hat{P}_j) = 1$, cf Eq.(26).
3. $\text{tr} \hat{\rho} = \frac{1}{d} \sum_j^{d^2} (\text{tr} \hat{\rho} \hat{P}_j) \text{tr} \hat{P}_j = \frac{1}{d} \sum_j^{d^2} (\text{tr} \hat{\rho} \hat{P}_j) = 1$.
4. $\text{tr} \hat{A} \hat{B} = \frac{1}{d} \sum_j^{d^2} (\text{tr} \hat{A} \hat{P}_j) (\text{tr} \hat{B} \hat{P}_j) = \frac{1}{d} \sum_j^{d^2} V_{AB}(j)$.
5. $\text{tr} \hat{A} \hat{B} = \frac{1}{d^2} \sum_j^{d^2} (\text{tr} \hat{A} \hat{P}_j) \sum_{j'}^{d^2} (\text{tr} \hat{B} \hat{P}_{j'}) \text{tr} \hat{P}_j \hat{P}_{j'} = \frac{1}{d} \sum_j^{d^2} V_A(j) V_B(j)$.

In finite dimensional studies it is convenient to use unitary operators [10, 18]. Thus the probability, given that the system is in the state $\hat{\rho}$, to measure it to be in the state $|m, b\rangle$ i.e. to be in an eigen function of XZ^b with eigenvalue ω^m without regard to any other probability is $\text{tr}(\hat{\rho} |m, b\rangle \langle b, m|)$. Here $|m, b\rangle$ is the eigen-function of the unitary operator XZ^b , cf Eq.(20). $\text{tr}(\rho |m, b\rangle \langle b, m|)$ is the finite dimensional observable that corresponds to $\text{tr}(\hat{\rho} |x', \theta\rangle \langle \theta, x'|)$ of Eq. (12) in the continuous case. Note that we may regard the c-number function gotten upon mapping, a la Wigner, the operator $\hat{\rho} |x', \theta\rangle \langle \theta, x'|$ onto phase space as a marginal quasi distribution of $W_\rho(q, p)$ cf. Eq.(11). In what follows we introduce, for the finite dimensional case, Wigner like mapping, now onto DAPG coordinates, that, correspondingly, relates its marginals to the full quasi distribution for the MUB state projectors.

Recalling, Eq.(24), $\hat{A}_\alpha \equiv |m, b\rangle \langle m, b|$, $\alpha = (m, b)$, we may write,

$$\text{tr} \hat{\rho} \hat{A}_\alpha = \frac{1}{d} \sum_j^{d^2} V_\rho(j) V_{A_\alpha}(j), \text{ cf. 4 above.} \quad (47)$$

Noting that,

$$V_{A_\alpha}(j) = \text{tr} A_\alpha P_j = \Lambda_{\alpha,j} \text{ cf. Eq.(27).} \quad (48)$$

i.e.

$$\text{tr} \hat{\rho} \hat{A}_\alpha = \frac{1}{d} \sum_{j \in \alpha} \text{tr} \hat{\rho} \hat{P}_j = \frac{1}{d} \sum_j \text{tr} \hat{\rho} \hat{P}_j \Lambda_{\alpha,j} = \frac{1}{d} \sum_j V_\rho(j) V_{A_\alpha}(j). \quad (49)$$

These equations correspond to Eqs.(11),(12) of Section II. Thus we identify $\text{tr} \rho A_\alpha$ as the (finite dimensional) Radon transform of $V_\rho(j)$ - it sums up the values of $V_\rho(j)$ for $j \in \alpha$. (The $\Lambda_{\alpha,j}$ plays the role of the delta function.) These equations correspond to Eq.(11, 10).

To obtain the (finite dimensional) inversion to the transform we consider,

$$\sum_{\alpha \in j'} \text{tr}(\rho \hat{A}_\alpha) = \frac{1}{d} \sum_j \text{tr} \rho P_j \sum_{\alpha \in j'} \Lambda_{\alpha,j} = \frac{1}{d} \sum_{j \in \alpha} \sum_{\alpha \in j'} \text{tr} \rho P_j = 1 + V_\rho(j). \quad (50)$$

Where in the last step we used the DAPG based relation,

$$\sum_{(\alpha \in j)} \sum_{(j' \in \alpha)} \hat{P}_j = \sum_{j'=1}^{d^2} \hat{P}_{j'} + d P_j = d \hat{I} + d \hat{P}_j.$$

Thence, the inversion is

$$V_\rho(j) = \sum_{\alpha \in j} \text{tr}(\rho \hat{A}_\alpha) - 1. \quad (51)$$

This could have been gotten directly from the definition of the operators however we could perhaps miss thereby some of the insight that the lengthy derivation provides which, in turn, underscores its relation with the continuous inverse Radon transform, Eq.(13).

VIII. AFFINE PLANE GEOMETRY (APG) FORMULATION

The central work in the underpinning of finite dimensional MUB operators with finite geometry, [1], is given in terms of lines and points of affine plane geometry (APG) rather than our choice of DAPG. An advantage of this, APG, scheme is its apparent similarity with the $d \rightarrow \infty$ case in that it involves square arrays and states projectors are straight lines (albeit in a modular sense) in a "discrete" phase space. It allowed a direct imposition of translational invariance and extension to dimensionality $d=p^n$; $n > 1$, p a prime, [9]. In this section we recast our DAPG underpinning into an APG one by interchanging lines and points and in particular the symmetrical meaning of $\Lambda_{\alpha,j}$, Eq. (27),(61) is shown to allow the *formulae* for the Radon transform remain intact.

Recall that within the DAPG underpinning a line was defined by the two points: $m(-1)$ and $m(0)$. The first, $m(-1)$, was associated with modulated position, as it relates to the eigen values of Z , viz the computational basis states. We shall refer to it by ξ . The second, $m(0)$, was associated with (modulated) momenta - it being the eigen values of X ($b=0$, Eq.(20), i.e. the Fourier transform of the former states. We shall refer to it by η . Thus a DAPG lines are $j = (m(-1) \equiv \xi, m(0) \equiv \eta)$; $\xi, \eta = 0, 1, 2, \dots, d-1$. Now consider a $d \cdot d$ square array, $d = p$, a prime $\neq 2$, whose (discrete) coordinates along the x axis is labelled by ξ and the y axis by η . We interpret each point, α , in the array, $\alpha = (\xi, \eta)$ as underpinning a DAPG line j . Thus the "image" of each DAPG line is a APG point. We now consider lines in this array:

$$\begin{aligned} \eta &= r\xi + s; \text{ mod}[d]; \quad r, s = 0, 1, 2, \dots, d-1. \\ \xi &= s'; \text{ mod}[d]; \quad s' = 0, 1, 2, \dots, d-1. \end{aligned} \quad (52)$$

Eq.(52) defines $d(d+1)$ "straight" lines: there are d^2 possibilities for r and s and d for s' . Each line contain d points: (ξ_i, η_i) , $i = 0, 1, \dots, d-1$. The line $\xi = s'$ contain d points as well: (s', η_i) , $i = 0, 1, \dots, d-1$. this proves APG(a), of section III. Similarly, the proofs of the validity of APG(x), $x=b, c, d$ and e for the lines, Eq.(??) and points forming the array are trivial. e.g. to prove APG(b): Consider two distinct lines, $\eta = r_1\xi + s_1$, $\eta = r_2\xi + s_2$, $r_1 \neq r_2$, $s_1 \neq s_2$. Let this lines share a point (ξ_0, η_0) . This implies, $r_1\xi_0 + s_1 = r_2\xi_0 + s_2$. This implies a *unique* (ξ_0, η_0) : $\eta_0 = r_1\xi_0 + s_1$; $\xi_0 = (s_1 - s_2)/(r_2 - r_1)$. For $r_1 = r_2$, $s_1 \neq s_2$, no common point exist (the lines are parallel). Thus the square array with points labelled by (ξ, η) and lines given by Eq(52) form a realization of APG. It is specified that a line $j = (m(-1), m(0)) \in \text{DAPG}$ is mapped to a point $(\xi, \eta) \in \text{APG}$. (Note that a point, α , of DAPG underpins an MUB projector: $\alpha = |m, b\rangle\langle b, m|$.)

Theorem: In a DAPG realization, the d lines of DAPG that form the image of the d points of a APG line, $\eta = r\xi + s$, share a point.

Proof: Consider an APG line, $\eta = r\xi + s$. It contain the d APG points $(\xi_i, \eta_i = r\xi_i + s)$, $i = 0, 1, \dots, d-1$. Now pick two arbitrary points i, i' . Their images in DAPG are the two lines $(m(-1) = \xi_i, m(0) = r\xi_i + s)$ and $(m(-1) = \xi_{i'}, m(0) = r\xi_{i'} + s)$. The equation for the point they share, cf Eq.(16), is

$$\begin{aligned} \frac{b}{2}(2\xi_i - 1) + r\xi_i + s &= \frac{b}{2}(2\xi_{i'} - 1) + r\xi_{i'} + s \text{ mod } [d], \\ \rightarrow (b+r)(\xi_i - \xi_{i'}) &= 0 \text{ mod } [d]. \end{aligned} \quad (53)$$

this is independent of $\xi_i - \xi_{i'}$. i.e. all the lines, $i, i'=1, 2, \dots, d-1$, share a common point at $b=-r \text{ mod } [d]$, thence the point is $m(b=-r)=r/2+s$. For the APG line $\xi = s'$ the common point within DAPG is, trivially, at $b=-1$: $m(-1)=s'$.

To illustrate the steps involved we consider $d=3$ with APG line given by $\eta = \xi + 1$. Thus the APG points involved are: $(0,1), (1,2)$ and $(2,0)$. Via Eq(35), Eq.(36) we have,

$$P_{(0,1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}; \quad P_{(1,2)} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}; \quad P_{(2,0)} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}. \quad (54)$$

Now, Eq.(53) relates these to the DAPG point (i.e. the MUB projector) $|0, 2\rangle\langle 2, 0| = A_{(0,2)}$

$$\frac{1}{3} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix} \right] = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad (55)$$

where the last matrix is $A_{(0,2)}$, cf. Eq.(34).

To distinguish between DAPG and APG underpinned operators we adopt the following scheme. The projector onto the MUB state $|m, b\rangle$, a point *alà* DAPG, was designated by $A_{(\alpha=m(b))}$ with the subscript indicating its coordinates. It is a line *alà* APG, and will be designated by $B_{(\lambda=\eta(\xi))}$ the subscript now gives the APG line's equation. The DAPG line operator P_j will, within APG underpinning scheme, be specified by the point $P_{(\xi,\eta)}$ giving its coordinates. Within this notation our Wigner function-like mapping function, Eq.(46), is,

$$\mathcal{V}_\rho(\xi, \eta) = \sqcup \nabla \rho Q_{(\xi,\eta)}. \quad (56)$$

Here the variables, ξ, η , signifies its APG underpinnings. In conformity with APG we have, (in correspondence with the DAPG relation, Eq.(53))

$$B_{(\eta=\eta(\xi))} = \frac{1}{d} \sum_{(\eta,\xi \in \beta)} Q_{(\eta,\xi)}. \quad (57)$$

For the sake of clarity we wish to iterate the relations between the *Hilbert space* operators P_j and $Q_{\xi,\eta}$: P_j is a line operator within DAPG underpinning, $j=(m(-1), m(0))$ - the line is fully parameterized by the "position" variable value, $m(-1)$ and the "momentum" $m(0)$. $Q_{\xi,\eta}$ is a APG underpinned point operator with ξ, η the point's coordinates with $\xi = m(-1), \eta = m(0)$. The APG underpinning was *constructed* by identifying,

$$P_{j=(m(-1), m(0))} \equiv Q_{(\xi=m(-1), \eta=m(0))}.$$

Thus although the subscript of P designate a *line* underpinning (DAPG); while that of Q, is a *point* underpinning (APG), they are equal when the subscripts are numerically equal.

The essential issue in the following is the simple, yet long winded, equivalence of our two accounts for the MUB operator, $|m, b\rangle\langle b, m|$:

(a) As a DAPG underpinned point, α . In this case $\alpha = m(b)$, specifies the coordinate of the point in the $d \cdot (d+1)$ points array. It is given by the point underpinned operator A_α .

(b) As a APG underpinned line, λ . In this case $\lambda = \eta(\xi)$ is the equation of the line whose constituent points, $\xi, \eta \in \lambda$, correspond to the DAPG underpinned lines parameterized with $m(-1)$, "position", and $m(0)$, "momentum" coordinate: $m(-1) \rightleftharpoons \xi$; $m(0) \rightleftharpoons \eta$. It is given by the line underpinned operator B_λ .

The mapping of the line operator, B_λ , is the Lambda function,

$$tr B_\lambda Q_{\xi,\eta} = \frac{1}{d} \sum_{\xi', \eta' \in \lambda} Q_{\xi', \eta'} Q_{\xi,\eta} = \Lambda_{((\xi', \eta'), \lambda)} = \begin{cases} 1, & (\xi', \eta') \in \lambda, \\ 0, & (\xi', \eta') \ni \lambda. \end{cases} \quad .QED. \quad (58)$$

i.e. it is non-vanishing only with the point (ξ, η) in the line λ . It is a straight forward matter to show that the mapping of the density operator onto APG phase space like lines and points, $\mathcal{V}_\rho(\xi, \eta)$, is a quasi distribution - i.e. it possess the equivalent attributes, 1 - 5, heeded by our DAPG mapping given in the last section. AS an example we prove item 5:

$$\begin{aligned} &= \frac{1}{d^2} \sum_{\xi, \eta} [tr A Q_{\xi,\eta}] \sum_{\xi', \eta'} [tr B Q_{\xi', \eta'}] tr Q_{\xi,\eta} Q_{\xi', \eta'} \\ &= \frac{1}{d} \sum_{\xi, \eta} (tr A Q_{\xi,\eta}) (tr B Q_{\xi,\eta}) = \frac{1}{d} \sum_{\xi, \eta} \mathcal{V}_A(\xi, \eta) \mathcal{V}_B(\xi, \eta). \end{aligned} \quad (59)$$

The Radon transform of the quasi distribution, $V_\rho(\xi, \eta)$, is $tr \rho B_\lambda$ - since in the present case the line underpinned operator is the MUB state projector. Thus,

$$tr \rho B_\lambda = \frac{1}{d} \sum_{\xi, \eta \in \lambda} tr \rho Q_{\xi,\eta} = \frac{1}{d} \sum_{\xi, \eta} (tr \rho Q_{\xi,\eta}) \Lambda_{(\xi,\eta), \lambda}. \quad (60)$$

This expression is in complete analogy with Radon transform in the continuum: we sum the quasi distribution over points (ξ, η) on the line λ . Given that the system is the state ρ , the probability to measure it to be in $|m, b\rangle$, is given by Eq.(60). This equation is analogous to Eq.(11): It expresses the probability of being in the MUB state (here $|m, b\rangle$) in terms of summation of the quasi distribution (here $\text{tr} \rho Q_{\xi, \eta}$) along a line determined by the mapping of the MUB projector onto the underpinning points (here $\Lambda_{(\xi, \eta), \lambda} = \text{tr} B_{\lambda} Q_{\xi, \eta}$).

IX. SUMMARY AND CONCLUDING REMARKS

The finite dimensional, d , density operator, and mutual unbiased basis (MUB) states' projectors were mapped onto points and lines of finite geometry. These mappings were shown to be analogous to the Wigner function mapping of the density operator and MUB state projectors in the continuum, $d \rightarrow \infty$. In the latter ($d \rightarrow \infty$) case, the expectation values of the MUB state projectors expressed via the appropriate Wigner function scheme were observed to be the Radon transform of the Wigner function of the state itself. Inverse Radon transform is, thus, state reconstruction in terms of the afore mentioned expectation values.

The proposed finite dimensional map of the density operator possess the quasi distributional attributes over its underpinning geometry as the Wigner function over phase space. The mapping of the MUB state projectors are, like their Wigner function counters, lines. These lines are arranged as straight lines for the affine plane geometry (APG) underpinning. The underpinning with the dual affine plane geometry (DAPG) which was most extensively employed allows a simpler formulation.

The expectation values of the MUB state projectors were used to define finite dimensional Radon transform. It involve summation along a line of the underpinning geometry. The inverse finite dimensional Radon transform is used for state reconstruction in close analogy with the continuum analysis.

A brief summary of dual affine plane geometry (DAPG) in finite dimension, d , was given. The geometry was used to underpin projectors of states of mutual unbiased bases (MUB) scanning a d -dimensional Hilbert space. The dimensionality studied were $d=p$, a prime, $\neq 2$.

The Wigner function, $W_{\rho}(q, p)$, may be viewed as a mapping of the density operator that act in Hilbert space onto a c -number function in phase space. A finite dimensional, d , Wigner function-like, $V_{\rho}(j)$, was defined a mapping of the finite dimensional density operator, ρ , onto c -number function of lines, $L_j = 1, 2, \dots, d^2$ and points, $\alpha = 1, 2, \dots, d(d+1)$, of dual affine plane geometry in d - dimensions. $V_{\rho}(j)$ posses all the attributes of $W_{\rho}(q, p)$ that qualify it as quasi distribution. A particularly attractive attribute of the Wigner function that underscores its role as quasi distribution is it marginals. In particular the marginals of mutual unbiased basis (MUB) state projector is itself a quasi distribution and is recognized as a Radon transform of $W_{\rho}(q, p)$. This transform involves integration along a line in phase space which relates to the MUB state projector map onto this space. In close analogy the marginal of a finite dimensional MUB state projector involves summation along a DAPG line and is itself a quasi distribution. This close analogy led, through its physical interpretation, to circumvent the angular involvement within Radon transformation and allowed a convenient definition of finite dimensional Radon transform concomitant with its inverse i.e. the state reconstruction.

In closing we present in a comparative way the mappings involved in this work. (The detailed meaning of the symbols are given in the text with the specified equations.)

(a) Continuum Hilbert space operator \hat{A} onto phase space (Wigner function). Here a "line" underpins an Mutual Unbiased Basis (MUB) state projector $|x', \theta\rangle\langle\theta, x'|$. Eq.(9).

(b) Finite dimension, $d=\text{prime} \neq 2$ Hilbert space operator \hat{A}_N onto points and lines of Dual Affine Plane Geometry (DAPG). Here a "point" $\alpha = m(b)$, underpins an MUB state projector $|m, b\rangle\langle b, m|$.

(c) Finite dimension, $d=\text{prime} \neq 2$ Hilbert space operator \hat{A}_N onto points and lines of Affine Plane Geometry (APG). Here a "line", $\lambda = (m(-1), m(0))$, underpins MUB state projector $|m, b\rangle\langle b, m|$:

$$(a) W_A(q, p) = \int dy e^{ipy} \langle q - y/2 | \hat{A} | q + y/2 \rangle. \text{ Eq.(9)}$$

$$(b) V_{A_N}(j) = \text{tr} \hat{A}_N P_j; j - \text{a line. Eq.(46).}$$

$$(c) \mathcal{V}_{A_N}(\xi, \eta) = \text{tr}(\hat{A}_N Q_{\xi, \eta}); (\xi, \eta) - \text{a point. Eq.(56).}$$

Likewise we give now the Radon transform of the density operator mappings.

$$(a) \text{tr}(\rho |x', \theta\rangle\langle\theta, x'|) = (\mathcal{R}W_{\rho})(x', \theta) = \int \frac{dq dp}{2\pi} \delta(x' - q \cos \theta - p \sin \theta) W_{\rho}(q, p). \delta(x' - q \cos \theta - p \sin \theta) = W_{|x' \theta\rangle\langle\theta, x'|}(q, p). \text{ Eq.(12).}$$

$$(b) \text{tr} \rho \hat{A}_\alpha = (\mathcal{R}_\mathcal{N} V_\rho)(\alpha = m(b)) = \frac{1}{d} \sum_j^{d^2} (\text{tr} \rho P_j) \Lambda_{\alpha,j}; \quad \Lambda_{\alpha,j} = V_{A_\alpha}(j). \quad \text{Eq.}(47).$$

$$(c) \text{tr} \rho \hat{B}_\lambda = (\mathcal{R}_\mathcal{N} \mathcal{V}_\rho)(\lambda = \eta(\xi)) = \frac{1}{d} \sum_{\xi,\eta} (\text{tr} \rho Q_{\xi,\eta}) \Lambda_{(\xi,\eta),\lambda}; \quad \Lambda_{(\xi,\eta),\lambda} = \mathcal{V}_{B_\lambda}(\xi, \eta). \quad \text{Eq.}(60).$$

Appendix A: The lambda function, $\Lambda_{(\alpha,j)}$

To prove the alternative meaning of the Lambda function, Eq.(61), consider

$$\text{tr} \hat{A}_\alpha \hat{P}_j = \frac{1}{d} \sum_{j' \in \alpha} \text{tr}(P_j P_{j'}) = \Lambda_{j,\alpha}.$$

Here we used the expression for A_α in terms of P_j , Eq.(22). Evaluating this gives,

$$\text{tr}\{\hat{A}_\alpha \hat{P}_j\} = \Lambda_{\alpha,j}; \quad \Lambda_{\alpha,j} = \begin{cases} 1, & j \in \alpha, \\ 0, & j \ni \alpha. \end{cases} \quad .QED. \quad (61)$$

Where we used Eqs.(37, 38).

Appendix B: Fluctuation Distillation Formula

Given, Eq(22), $\hat{P}_j = \sum_{\alpha \in j} \hat{A}_\alpha - \hat{I}$ and, Eq(37), $\hat{P}_j^2 = \hat{I}$, implies

$$\left(\sum_{\alpha \in j} \hat{A}_\alpha - \hat{I} \right) \left(\sum_{\alpha' \in j} \hat{A}_{\alpha'} - \hat{I} \right) = \hat{I}.$$

Thus,

$$\sum_{\alpha, \alpha' \in j} \hat{A}_\alpha \hat{A}_{\alpha'} = 2 \sum_{\alpha \in j} \hat{A}_\alpha.$$

Recalling that, Eq(24), $A_\alpha^2 = A_\alpha$ allows

$$\sum_{\alpha \neq \alpha' \in j} \hat{A}_\alpha \hat{A}_{\alpha'} = \sum_{\alpha \in j} \hat{A}_\alpha.$$

QED

Appendix C: The relation $\hat{P}_{j=m(-1),m(0)}^2 = \hat{I}$

Recalling Eqns.(30, 31, 36) we have

$$(\hat{P}_{j=c/2,m(0)})_{n,n'} = \begin{cases} \omega^{-(n-n')m(0)} \delta_{(n+n'),c} & \\ 0 & \text{otherwise} \end{cases} \quad (62)$$

Squaring this matrix we have trivially 1 at $n=n'$, $n=c/2$. The only non nil element of the n -th row ($n \neq c/2$) of the matrix is along the column $n'=c-n$, where it is given by $\omega^{(n'-n)m(0)}$. The only column having non nil element at the row $n=c-n'$ is the n -th column with the element $\omega^{(n-n')m(0)}$. Thus $(\hat{P}_{j=c/2,m(0)}^2)_{n,n'} = \delta_{n,n'}$ QED.

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